Department of Mathematics

MAL 111 (2012 October) Minor Test 2

Time: 1 hour **Maximum Marks: 22**

Every problem is compulsory. No marks will be awarded if appropriate arguments are not provided while answering the questions.

(1) Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function (with the usual metric on $\mathbb{R}$ and on its subset $[a, b]$). Show by using the concept of compactness and connectedness that $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$. [4]

4

(2) Suppose $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are both continuous with respect to the usual metric. Consider $\mathbb{R}^2$ with the metric defined by [3 + 2 = 5]

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

3½

(a) Show by $\epsilon - \delta$ definition that the map $\phi : \mathbb{R} \to \mathbb{R}^2$ defined by $\phi(x) = (h_1(x), h_2(x))$ is continuous.

(b) Show that the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a connected subset of $\mathbb{R}^2$.

$\mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

(3) Compute $R_2(x)$ and $R_3(x)$ in the Taylor's formula for the function $\tan^{-1} : \cancel{(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}}$ about $x = 0$. Use it to show that there exists $\delta > 0$ such that for every $x \in [0, \delta]$, we have [4]

3½

$$x - \frac{x^3}{3} \leq \tan^{-1}(x) \leq x.$$

(4) Show that the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by [3]

3

$$h(x, y) = \begin{cases} \frac{x^4 - y^2}{x^4 + y^2} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0), \end{cases}$$

is not continuous.

(5) (a) Find the following limit. [3]

3

$$\lim_{x \to \infty} x^{\frac{1}{x}}.$$

(b) Find $f_x(0, b)$ for every $b \in \mathbb{R}$ if [3]

3

$$f(x, y) = \begin{cases} \frac{y^2}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (0, 0) = (0, 0). \end{cases}$$

::: **END** :::